

Influence of External Fields and Environment on the Dynamics of Phase Qubit-Resonator System

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Abstract

We analyze the dynamics of a qubit-resonator system coupled with a thermal bath and external electromagnetic fields. Using the evolution equations for the set of Heisenberg operators, that describe the whole system, we derive an expression for the resonator field, accounting for the resonator-drive, -bath, and -qubit interaction. The renormalization of the resonator frequency, caused by the qubit-resonator interaction, is accounted for. Using solutions for the resonator field, we derive the equation describing qubit dynamics. The influence of the qubit evolution during the measurement time on the fidelity of a single-shot measurement is studied. The relation between the fidelity and measurement time is shown explicitly. Also, an expression

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describing relaxation of the superposition qubit state towards its stationary value is derived. The possibility of controlling this state, by varying the amplitude and frequency of drive, is shown.

1 Introduction

The possibility of achieving quantum coherence in macroscopic Josephson junction (JJ) circuits, envisioned by Leggett in the early 1980s [1]-[3], was demonstrated experimentally almost 20 years later by Nakamura *et al.* [4]. Now, superconductive qubits are promising building blocks for the realization of a quantum computer. Progress towards quantum computing depends on the development of measurement schemes and readout devices. Each read-out process requires a finite-time interval during which the system evolves under the effect of the measuring device [5]-[7] and uncontrollable external [8],[9] or intrinsic [10]-[14] noises. This requires improving the quality of the Josephson contacts and effective isolation of qubit systems from the “electromagnetic environment”. There is stable progress toward the solution of the first problems. (See, for example, Ref. [15].) The second problem is fundamental in nature because, for example, it is impossible to isolate the qubit system from vacuum zero-point oscillations or to avoid the influence of the measurement device on qubit relaxation or decoherence.

In this paper, we study the simultaneous influence of external electromagnetic fields and a thermostat (bath) on the qubit. It is assumed that this influence is not direct but via a resonator which is weakly coupled with the qubit. Therefore, the resonator response on the microwave field depends on the qubit state. In particular, by measuring the phase of a microwave field reflected from the resonator, one can perform a nondemolition measurement of the qubit state. The theoretical description of this scheme and its practical implementation are elucidated in numerous publications [5]-[7], [16]-[20]. At the same time, it should be emphasized that this kind of measurement results in both the dephasing of the qubit wave function and partial relaxation caused by the effects of drive and the thermostat during the measurement. These effects decrease the measurement fidelity. By studying the qubit dynamics, we are able to estimate the influence of drive and bath on the fidelity.

Our further consideration is based on the equations of motion of the Heisenberg operators. These operators describe the dynamics of both the

qubit and resonator field. The solution of these dynamical equations will allow us to estimate the upper limit of the measurement fidelity.

2 Model

We consider the system described by the following Hamiltonian:

$$H = -\frac{1}{2}\hbar\omega_q\sigma_z + \hbar\omega_r\left(a^+a + \frac{1}{2}\right) + \sum_n \hbar\omega_n\left(b_n^+b_n + \frac{1}{2}\right) + ig\sigma_y\left(a^+ - a\right) + i\hbar\sum_n f_n\left(b_na^+ - b_n^+a\right) + i\hbar f_0\left(ca^+ - c^+a\right), \quad (1)$$

where the first, second, and third terms in the right side are Hamiltonians of the noninteracting qubit, the resonator, and the bath, respectively. The quantity, σ_z , is the Pauli operator. We use here the diagonal representation for Hamiltonian of the isolated qubit. Different types of superconducting qubits, for example, charge qubit [5]-[7] or flux-biased phase qubit [13] can be used in practice.

The quantities a^+, b_n^+, c^+ are the creation operators of the corresponding excitations and a, b_n, c are the annihilation operators. Usually, the drive variables, c^+ and c , are considered to be the classical quantities, c^* and c , with given dependences on time: $c^*, c \sim e^{\pm i\omega_d t}$ in which ω_d is the frequency of microwave field.

The fourth, fifth, and sixth terms are the resonator-qubit, -bath, and -drive interactions, respectively. The constants g , f_n , and f_0 , describe the corresponding interaction strengths. ω_q is the transition frequency between qubit levels. The quantities, ω_r and ω_n , are the frequencies of the resonator (cavity) and bath oscillators, respectively. The resonator Hamiltonian describes the parallel connection of the local capacitance, C , and the inductance, L ($\omega_r = (LC)^{-1/2}$). At the same time this lumped-element circuit can represent a transmission (microstrip) line if we deal with signals whose characteristic frequencies are sufficiently close to one of the eigenfrequencies, ω_r , of the line. More details can be obtained, for example, in [5] and [22].

Similar to papers [21],[22], we model the bath as an infinite set of harmonic oscillators with frequencies, ω_n . It can be seen from the explicit resonator-bath interaction Hamiltonian that the presence of the bath is important even at zero temperature. In this case, only terms describing the annihilation of resonator excitations and the creation of bath excitations

appear. At finite bath temperatures, there are fluxes of energy in both directions: into and out of the bath. There is no unique description of the effect of the bath on the qubit-resonator system. For example, the model of a direct qubit-bath interaction is used in Ref. [23].

The resonator-bath as well as the resonator-drive interactions affect the qubit state due to the qubit-resonator interaction. This situation is very similar to the case considered in [8] in which semiclassical noise of the biased current causes qubit decoherence and relaxation.

3 Evolution of the resonator field

Using the Heisenberg representation for the operators in Eq. 1, we can express the time derivative of a as

$$\dot{a} = \frac{1}{i\hbar}[a, H] = -i\omega_r a + \frac{g}{\hbar}\sigma_y + \sum_n f_n b_n + f_0 c. \quad (2)$$

Here, the dependences of σ_y and b_n on time are not yet specified. To determine the explicit dependences of $\sigma_y(t)$ and $b_n(t)$, we will use an iterative procedure that assumes that all interaction parameters are small quantities. As in Eq. 2 we have

$$\begin{aligned} \dot{\sigma}_y &= -\omega_q \sigma_x, \\ \dot{\sigma}_x &= \omega_q \sigma_y + i2\frac{g}{\hbar}\sigma_z(a^+ - a). \end{aligned} \quad (3)$$

From these equations we can obtain relationships for the more convenient variables $\sigma_{\pm} \equiv \frac{1}{2}(\sigma_x \pm i\sigma_y)$:

$$\dot{\sigma}_{\pm} = \mp i\omega_q \sigma_{\pm} + i\frac{g}{\hbar}\sigma_z(a^+ - a). \quad (4)$$

Eqs. 4 can be rewritten in the equivalent form as

$$\sigma_{\pm}(t) = \sigma_{\pm}(t_0)e^{\mp i\omega_q(t-t_0)} + i\frac{g}{\hbar}\int_{t_0}^t dt' e^{\mp i\omega_q(t-t')}[\sigma_z(a^+ - a)]_{t'}. \quad (5)$$

Further analysis is facilitated by assuming the qubit-resonator detuning is small:

$$|\omega_{qr}| \ll \omega_q, \omega_r, \quad \omega_{qr} \equiv \omega_q - \omega_r.$$

(We assume this inequality is satisfied throughout the paper.) At the same time the detuning should be large compared to the interaction parameter g :

$$\frac{g}{\hbar|\omega_{qr}|} \ll 1. \quad (6)$$

In many papers (see, for example, Refs. [5]-[7]), the quantity in the left side of Eq. 6 is also considered as small parameter. In this case, the dynamics of the system can be analysed using a simpler (renormalized) Hamiltonian.

Considering $\sigma_z(t')a^+(t')e^{-i\omega_r t'}$ and $\sigma_z(t')a(t')e^{i\omega_r t'}$ as slowly varying functions of t' , we can integrate over t' in Eq. 5. The result is

$$\begin{aligned} \sigma_+(t) &= \tilde{\sigma}_+(t) - \frac{g}{\hbar} \frac{\sigma_z(t)a(t)}{\omega_{qr}}, \\ \sigma_-(t) &= \tilde{\sigma}_-(t) - \frac{g}{\hbar} \frac{\sigma_z(t)a^+(t)}{\omega_{qr}}, \end{aligned} \quad (7)$$

where $\tilde{\sigma}_\pm = \sigma_\pm(t_0)e^{\mp i\omega_q(t-t_0)}$. In the course of integration, we have assumed that $t - t_0 \rightarrow \infty$.

Using Eqs. 7 we can express $\sigma_y(t)$, which appears in Eq. 2, in terms of $\tilde{\sigma}_\pm(t)$, $\sigma_z(t)$, $a^+(t)$, and $a(t)$. The dependences of b_n^+ , b_n on time can be also expressed in terms of a^+ and a . As in Eqs. 4, we have

$$\dot{b}_n = -i\omega_n b_n - f_n a, \quad (8)$$

or

$$b_n(t) = \tilde{b}_n(t) - f_n \int_{t_0}^t dt' e^{-i\omega_n(t-t')} a(t'), \quad (9)$$

where $\tilde{b}_n(t) = b_n(t_0)e^{-i\omega_n(t-t_0)}$.

Multiplying both sides of Eq. 9 by f_n and summing over n we obtain

$$\sum_n f_n b_n(t) = \sum_n f_n \tilde{b}_n(t) - \int_{t_0}^t dt' \sum_n f_n^2 e^{-i\omega_n(t-t')} a(t'). \quad (10)$$

Using a simple approximation for the sum in the integrand of Eq. 10

$$\sum_n f_n^2 e^{-i\omega_n(t-t')} = \kappa \delta(t - t'), \quad (11)$$

(see Refs. [21],[22]), we can rewrite Eq. 10 in a simple form:

$$\sum_n f_n b_n(t) = \sum_n f_n \tilde{b}_n(t) - \frac{\kappa}{2} a(t). \quad (12)$$

Then using Eqs. 7 and 12, the equation for $a(t)$ reduces to

$$\left[\partial_t + i \left(\omega_r - \chi \sigma_z \right) + \frac{\kappa}{2} \right] a = f_0 c + \sum_n f_n \tilde{b}_n - i \frac{g}{\hbar} \left(\tilde{\sigma}_+ - \tilde{\sigma}_- \right) - i \chi \sigma_z a^+, \quad (13)$$

where $\chi \equiv g^2 / (\hbar^2 \omega_{qr})$.

The quantity, $\omega_r - \chi \sigma_z$, is the resonator frequency renormalized by the qubit-resonator interaction. This renormalization effect can be derived from the Jaynes-Cummings Hamiltonian using a unitary transformation, assuming that $g / (\hbar \omega_{qr})$ is a small parameter.

The parameter, $\kappa/2$, appearing in the bath-resonator interaction, is an important characteristic of the resonator. It can be seen from the structure of Eq. 13 that $\kappa/2$ describes the field dissipation caused by this interaction. The ratio ω_r / κ is the resonator quality factor, Q . Usually, high- Q resonators are used for qubit measurements.

The solution of Eq. 13 (in which the influence of the initial condition or transient stage is ignored) is given by:

$$a(t) = \frac{if_0 c(t)}{\tilde{\omega}_{dr} + i\kappa/2} + \sum_n \frac{if_n \tilde{b}_n(t)}{\tilde{\omega}_{nr} + i\kappa/2} + \frac{g}{\hbar} \frac{\tilde{\sigma}_+(t)}{\tilde{\omega}_{qr} + i\kappa/2}, \quad (14)$$

where $\tilde{\omega}_{ir} \equiv \omega_i - \omega_r + \chi \sigma_z$, $i = d, n, q$. In the course of solution of Eq. 13, explicit dependences of $c, \tilde{b}_n, \tilde{\sigma}_\pm$ on t were used. Also, the contribution of terms with σ_- and a^+ was neglected. This approximation is accurate when $\chi, (g/\hbar) \ll \omega_r$.

The value for a^+ can be obtained from Eq. 14 using hermitian conjugation. It is given by

$$a^+(t) = \frac{if_0^* c^+(t)}{\tilde{\omega}_{rd} + i\kappa/2} + \sum_n \frac{if_n^* \tilde{b}_n^+(t)}{\tilde{\omega}_{rn} + i\kappa/2} - \frac{g}{\hbar} \frac{\tilde{\sigma}_-(t)}{\tilde{\omega}_{rq} + i\kappa/2}. \quad (15)$$

Using Eqs. 14, 15, and 11, we can easily show that the standard commutation relations between operators a and a^+ are fulfilled with an accuracy valid up to a small value of the order $g^2 / (\hbar^2 \omega_{qr}^2)$ if the drive variables are considered as classical quantities. The deviation of $[a, a^+]$ from unity is within the accuracy of perturbation procedure used here.

It follows from Eqs. 14 and 15 that the effect of bath is represented by the second (“noise”) terms and the imaginary summand, $i\kappa/2$, in the denominators. The effect of drive on the resonator field critically depends

on the detuning $\tilde{\omega}_{dr} = \omega_d - \omega_r + \chi\sigma_z$. The field amplitude and the photon number in the cavity, $n_r = a^\dagger a$, are the largest when the detuning is of the order of $k/2$. If the drive frequency is fixed, the detuning depends on the qubit state. For the upper and lower states, the resonant conditions can be very different when $\chi > \kappa/2$. This circumstance is commonly used for measuring qubit states by means of microwave fields (see, for example, Ref. [17]).

To estimate the importance of the different terms in Eqs. 14 and 15, we will calculate the average (over bath variables) photon number in the resonator, n_r^b , considering the qubit to be in the excited ($\sigma_z = -1$) or ground ($\sigma_z = 1$) state. (The frequency, $\tilde{\omega}_{dr}$, is equal to $\omega_d - \omega_r \mp \chi$ for $\sigma_z = \mp 1$, respectively.) Using Eq. 11 and the relationship

$$\sigma_- \sigma_+ = \frac{1}{2}(1 - \sigma_z), \quad (16)$$

we obtain

$$n_r^b = \frac{|f_0 c|^2}{\tilde{\omega}_{dr}^2 + \kappa^2/4} + \langle b_n^\dagger b_n \rangle_{\omega_n=\omega_r} + \frac{g^2}{2\hbar^2} \frac{1 - \sigma_z}{\tilde{\omega}_{qr}^2}, \quad (17)$$

where correlations between different bath modes were ignored ($\langle b_n^\dagger b_{n'} \rangle \sim \delta_{n,n'}$). In the course of derivation of the second term in Eq. 17, we have considered that the average $\langle b_n^\dagger b_n \rangle$ depends on n via ω_n only. Then using Eq. 11 we were able to sum up over bath modes as:

$$\begin{aligned} \sum_n \frac{f_n^2 \langle b_n^\dagger b_n \rangle}{\tilde{\omega}_{nr}^2 + \kappa^2/4} &= \sum_n \int_{-\infty}^{\infty} d\omega \delta(\omega - \tilde{\omega}_{nr}) \frac{f_n^2 \langle b_n^\dagger b_n \rangle|_{\omega_n=\omega+\omega_r-\chi\sigma_z}}{\omega^2 + \kappa^2/4} \approx \\ &\langle b_n^\dagger b_n \rangle|_{\omega_n=\omega_r} \int_{-\infty}^{\infty} \frac{d\omega}{\omega^2 + \kappa^2/4} \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} \sum_n f_n^2 e^{i(\omega-\tilde{\omega}_{nr})\tau} = \langle b_n^\dagger b_n \rangle|_{\omega_n=\omega_r}. \end{aligned}$$

The remarkable peculiarity of Eq. 17 is that the direct contribution of bath does not depend on the interaction constant, f_n . This is in contrast to Eqs. 14 and 15 for the resonator field.

For thermal equilibrium, the average occupancy of the bath modes is given by the Bose-Einstein distribution function n_{BE}

$$\langle b_n^\dagger b_n \rangle = n_{BE}(\omega_n) \equiv \left(e^{\frac{\hbar\omega_n}{k_B T}} - 1 \right)^{-1} \quad (18)$$

Therefore, in the absence of the drive and the qubit, the number of photons in the resonator is equal to that in the corresponding bath mode. In other

words, the bath and resonator temperatures are equal in this (equilibrium) case.

Eq. 17 shows explicitly when the driving field dominates the noisy influence of the bath. Besides that, it follows from Eq. 17 that the qubit “delivers” an almost negligible portion of photons to the resonator (the last term in Eq. 17) even in the most favorable case, $\sigma_z = -1$. The physical reason for this is in the qubit-resonator detuning. The detuning decreases the probability of qubit excitations to “penetrate” into the cavity.

Eqs. 14 and 15 can be used to study the fluctuations of photon numbers in the resonator. These fluctuations are responsible for qubit decoherence (more details can be obtained, for example, in Ref. [6]).

In the next Section, we will use Eqs. 14 and 15 to describe the qubit dynamics. Drive- and thermostat-induced variations of the qubit state during the measurement time will be studied. These variations are responsible for reducing the measurement fidelity of the qubit state.

4 Qubit evolution

The time variation of qubit states occupancies can be expressed in terms of the average value of the operator σ_z . When the qubit is in the state $\psi(t) = \alpha(t)|0\rangle + \beta(t)|1\rangle$, the average value of σ_z is:

$$\langle\sigma_z\rangle_t = \langle\psi(t)|\sigma_z|\psi(t)\rangle = |\alpha(t)|^2 - |\beta(t)|^2.$$

If we know $\langle\sigma_z\rangle_t$, the occupancies of the levels can be obtained from:

$$|\alpha(t)|^2 = \frac{1}{2}(1 + \langle\sigma_z\rangle_t), \quad |\beta(t)|^2 = \frac{1}{2}(1 - \langle\sigma_z\rangle_t).$$

We will replace $\langle\sigma_z\rangle_t$ by $\langle\sigma_z(t)\rangle$, where $\sigma_z(t)$ is defined in the Heisenberg representation and averaging is over the initial state, $\psi(t_0)$. Therefore, the evolution of the occupancies can be obtained from the temporal dependence of the operator $\sigma_z(t)$.

To obtain $\sigma_z(t)$, we use, as previously, the equations of motion for the corresponding operators. The time variation for $\sigma_z(t)$ is:

$$\dot{\sigma}_z = \frac{1}{i\hbar}[\sigma_z, ig\sigma_y(a^+ - a)]. \quad (19)$$

We can express $\sigma_y(a^+ - a)$ as $-i(\sigma_+a^+ + \sigma_-a)$ in the spirit of rotating-wave approximation. Then Eq. 19 reduces to

$$\dot{\sigma}_z \approx -i\frac{2g}{\hbar}(\sigma_+a^+ - a\sigma_-). \quad (20)$$

In the next step, we will express the right-hand side of Eq. 20 in terms of $\sigma_z(t)$. Similar to the consideration in the previous Section, we use the equation of motion for the operators σ_+a^+ , σ_-a . Thus we have

$$(\partial_t + i\omega_{qr})\sigma_+a^+ = i\frac{g}{\hbar}\sigma_z a^+(a^+ - a) + i\frac{g}{2\hbar}(1 - \sigma_z) + \sigma_+\left(f_0c^+ + \sum_n f_n b_n^+\right). \quad (21)$$

Using Eq. 12 and neglecting the term containing a^+a^+ , we can rewrite Eq. 21 as

$$\left(\partial_t + i\omega_{qr} + \frac{\kappa}{2}\right)\sigma_+a^+ = -i\frac{g}{\hbar}\sigma_z\left(a^+a + \frac{1}{2}\right) + i\frac{g}{2\hbar} + \sigma_+\left(f_0c^+ + \sum_n f_n \tilde{b}_n^+\right). \quad (22)$$

Using Eqs. 7 and 14, we obtain from Eq. 22

$$\begin{aligned} \left(\partial_t + i\omega_{qr} + \frac{\kappa}{2}\right)\sigma_+a^+ &= \tilde{\sigma}_+\left(f_0c^+ + \sum_n f_n \tilde{b}_n^+\right) - i\frac{g}{\hbar}\left[\sigma_z\left(a^+a + \frac{1}{2}\right) - \frac{1}{2}\right] + \\ &\frac{g\sigma_z}{\hbar\omega_{qr}}\left(\frac{f_0c}{\tilde{\omega}_{dr} + i\kappa/2} + \sum_n \frac{f_n \tilde{b}_n}{\tilde{\omega}_{nr} + i\kappa/2}\right)\left(f_0c^+ + \sum_{n'} f_{n'} \tilde{b}_{n'}^+\right). \end{aligned} \quad (23)$$

Let us average both sides of Eq. 23 over the thermostat variables. Then again considering σ_z as a slowly varying function of time, we can easily obtain $\langle\sigma_+a^+\rangle_{bath}$ from Eq. 23 in the form

$$\begin{aligned} \langle\sigma_+a^+\rangle_{bath} &= \frac{ig\left(e^{(-i\omega_{qr}-\kappa/2)t} - 1\right)}{\hbar(i\omega_{qr} + \kappa/2)}\left\{\left[\sigma_z\left(n_r^b + \frac{1}{2}\right) - \frac{1}{2}\right] + \right. \\ &\left. i\frac{\sigma_z}{\omega_{qr}}\left(\frac{|f_0c|^2}{\tilde{\omega}_{dr} + i\kappa/2} + \sum_n \frac{f_n^2\langle b_n b_n^+\rangle}{\tilde{\omega}_{nr} + i\kappa/2}\right)\right\}, \end{aligned} \quad (24)$$

where we have neglected the initial (at $t = 0$) correlations of the operator σ_+ with the operators a^+, c^+ . The value of n_r^b is given by Eq. 17 in which the last term, representing the contribution of qubit, can be omitted due to its small value.

The expression for $\langle a\sigma_z \rangle_{bath}$ can be derived in a similar manner. Then the equation for σ_z is:

$$\dot{\sigma}_z = \frac{2g^2}{\hbar^2\omega_{qr}^2} \left\{ \left[2\omega_{qr}\sin(\omega_{qr}t)e^{-\frac{\kappa}{2}t} + \kappa \left(1 - \cos(\omega_{qr}t)e^{-\frac{\kappa}{2}t} \right) \right] \left[\frac{1}{2} - \sigma_z \left(n_r^b + \frac{1}{2} \right) \right] - \sigma_z \frac{|f_0c|^2}{\tilde{\omega}_{dr}^2 + \kappa^2/4} \left[2\tilde{\omega}_{dr} \left(1 - \cos(\omega_{qr}t)e^{-\frac{\kappa}{2}t} \right) + \kappa \sin(\omega_{qr}t)e^{-\frac{\kappa}{2}t} \right] \right\}. \quad (25)$$

Because $\omega_{qr} \gg \kappa/2$, we can omit in Eq. 25 the oscillating terms that are proportional to κ and $\tilde{\omega}_{dr}$. Thus, the rate equation reduces to

$$\dot{\sigma}_z = \frac{2g^2}{\hbar^2\omega_{qr}^2} \left\{ \left[2\omega_{qr}\sin(\omega_{qr}t)e^{-\frac{\kappa}{2}t} + \kappa \right] \left[\frac{1}{2} - \sigma_z \left(n_r^b + \frac{1}{2} \right) \right] - \sigma_z \frac{2\tilde{\omega}_{dr}|f_0c|^2}{\tilde{\omega}_{dr}^2 + \kappa^2/4} \right\}. \quad (26)$$

In the absence of drive ($f_0 = 0$), Eq. 26 describes small-amplitude thermostat-induced Rabi oscillations with frequency ω_{qr} . These oscillations decay during the coherence time, $\sim 2/\kappa$, of the resonator field. The oscillations are accompanied by a slow qubit relaxation to the stationary value

$$\sigma_{st} = \left(1 + 2\langle b_n^+ b_n \rangle|_{\omega_n=\omega_r} \right)^{-1}. \quad (27)$$

(See, for example, Ref. [24].) The amplitude of oscillations depends on: the occupancies of the bath states, $\langle b_n^+ b_n \rangle|_{\omega_n=\omega_r}$, the cavity losses, $\kappa/2$, and the qubit-resonator detuning, ω_{qr} . In the case of an equilibrium bath, σ_{st} coincides with the equilibrium value known in the literature. It can be derived in an alternative manner using the density matrix formalism.

The case of $f_0 \neq 0$ is more interesting in view of the possibility of using the drive to measure (control) the qubit. The preferred setup is realized if the microwave field is in resonance with the cavity-qubit system in which the qubit is in a given eigenstate (for example, in the excited state: $\omega_d = \omega_r + \chi$). (This is the case for which the last term in braces of Eq. 26 vanishes.) The measuring device can resolve the resonator frequencies $\omega_r \pm \chi$, corresponding to the different eigenstates of the qubit, only if the measurement time, τ , is greater than $(2\chi)^{-1}$. At the same time, τ should be as small as possible to provide rapid control. Moreover, during the measurement time, the qubit evolves as described by the rate equation (26), thus decreasing the measurement fidelity F . As a consequence of the measurement, an arbitrary superposition qubit state collapses to one of the eigenstates with $\sigma_z = \pm 1$.

The characteristic time of the collapse (defined here as the decoherence time) is assumed to be considerably shorter than τ . Therefore, the fidelity of the measurement, F , for the qubit in the post-collapse state, $|1\rangle$, is defined as

$$F = \frac{1}{\tau} \int_0^\tau dt |\beta(t)| = \frac{1}{\tau} \int_0^\tau dt \left(\frac{1 - \sigma_z}{2} \right)^{1/2} \approx 1 - \frac{g^2(n_r^b + 1)}{\hbar^2 \omega_{qr}^2} \left(1 - \frac{\sin(\omega_{qr}\tau)}{\omega_{qr}\tau} \right). \quad (28)$$

For simplicity, we have considered the case $\tau < 2/\kappa$ in which σ_z is given by

$$\sigma_z(t) \approx -1 + \frac{8g^2(n_r^b + 1)}{\hbar^2 \omega_{qr}^2} \sin^2\left(\frac{\omega_{qr}t}{2}\right).$$

Taking into account $\omega_{qr}\tau \gg 1$, we have

$$F \approx 1 - \frac{g^2(n_r^b + 1)}{\hbar^2 \omega_{qr}^2}. \quad (29)$$

This formula illustrates the effect of the Rabi oscillations, generated by the external drive and bath, on the fidelity. In contrast to $\sigma_z(t)$, the fidelity does not display the oscillating behavior. This is because the value of σ_z is averaged over the interval $\tau \gg |\omega_{qr}|^{-1}$. For longer measurement times, $\tau > 2/\kappa$, all terms in Eq. 26 should be used to calculate $\sigma_z(t)$ and $F(\tau)$.

It seems from Eq. 29 that the fidelity can be improved for smaller interaction parameters. But decreasing g will decrease $\chi = g^2/(\hbar^2 \omega_{qr})$. In view of the inequality $\tau > (2\chi)^{-1}$ and using Eq. 29, we obtain

$$\tau > \frac{n_r^b + 1}{2(1 - F)\omega_{qr}}. \quad (30)$$

It follows from Eq. 30 that as F approaches unity, the measurement time should be increased.

We have described theoretically a single-shot measurement in which the initial qubit was assumed to be in either the excited or the ground state. The evolution of the superposed state can be investigated experimentally by the repetition of many single-shot measurements. The rate equation for $\langle \sigma_z(t) \rangle$ [the averaging is over the initial state, $\psi(t=0)$], corresponding to this kind of the experiment, can be derived in a manner similar to Eq. 26. It is given by

$$\langle \dot{\sigma}_z \rangle = \frac{\kappa g^2}{\hbar^2 \omega_{qr}^2} \left[-\langle \sigma_z \rangle (\varphi^+ + \varphi^-) + \varphi^- - \varphi^+ + 1 \right], \quad (31)$$

where the initial (oscillatory) stage of evolution is ignored;

$$\varphi^\pm \equiv \varphi(\sigma_z = \pm 1), \quad \varphi(\sigma_z) = n_r^b + \frac{1}{2} + \frac{2\tilde{\omega}_{dr}}{\kappa} \frac{|f_0 c|^2}{\tilde{\omega}_{dr}^2 + \kappa^2/4}. \quad (32)$$

In the derivation of Eq. 31 the following identities were used:

$$\varphi(\sigma_z) = \frac{1}{2}[(1 + \sigma_z)\varphi^+ + (1 - \sigma_z)\varphi^-].$$

and

$$\sigma_z \varphi(\sigma_z) = \frac{1}{2}[(1 + \sigma_z)\varphi^+ - (1 - \sigma_z)\varphi^-]. \quad (33)$$

The solution of Eq. 31 is given by

$$\langle \sigma_z(t) \rangle = \sigma_{st} + \left(\langle \sigma_z(t=0) \rangle - \sigma_{st} \right) e^{-\gamma t}, \quad (34)$$

in which the stationary value, σ_{st} , and the relaxation constant, γ , are given by

$$\sigma_{st} = \frac{\varphi^- - \varphi^+ + 1}{\varphi^- + \varphi^+}, \quad \gamma = \frac{k g^2 (\varphi^- + \varphi^+)}{\hbar^2 \omega_{qr}^2}. \quad (35)$$

In the limiting case $f_0 = 0$, σ_{st} reduces to the previous result given by Eq. 27. In the limit of dominating drive in Eq. 17 ($n_r^b \gg 1$ for the resonance conditions, $\omega_d = \omega_r \pm \chi$), the qubit relaxes to the ground or excited states, respectively. In the case of large resonator-drive detuning, $|\omega_{dr}| \gg \chi$, the last term in Eq. 32 can dominate. Then $\langle \sigma_z \rangle$ relaxes to zero.

5 Conclusion

We have described a qubit-resonator system with an external drive and thermostat. Our consideration is based on the equations of motion of the operators in the Heisenberg representation. This is in contrast to the widely used density matrix approach. Considering the resonator-bath, resonator-drive, and resonator-qubit interactions as weak perturbations, we have derived expressions for the resonator field including the renormalization of the resonator frequency caused by the qubit-resonator interaction. A weak qubit-environment interaction is a necessary condition for reliable isolation of the qubit from the “external world.”

Also, we have derived the rate equation for the qubit variable, σ_z , describing the occupancies of the qubit levels. The solution of this rate equation enables us to calculate the measurement fidelity, F , and to determine the dependence of F on the measurement time, τ : increasing fidelity requires increasing the measurement time. (See Eq. 30.) Both quantities are very important parameters in view of the practical implementations of qubits. Therefore, the optimal choice of F and τ should be carried out considering their interdependency given by inequality 30.

The qubit relaxation, caused by the interactions with the bath and the drive, can be used to control the final qubit state. In particular, it follows from Eqs. 31-35 that, by varying the frequency and amplitude of the drive as well as the interaction time, t , we can get a qubit with a predetermined probabilities to be in the ground or excited state.

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